

Electrical Networks with Prescribed Current and Applications to Random Walks on Graphs

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Abstract

We study the inverse problem of determining the conductivity matrix of an electrical network from the prescribed knowledge of the magnitude of the induced current along the edges coupled with the imposed voltage or injected current on the boundary nodes. This problem leads to a weighted l^1 minimization problem for the corresponding voltage potential. We also investigate the problem of determining the transition probabilities of random walks on graphs from the prescribed net number of times the walker passes along the edges of the graph. We also show that a mass preserving flow $J = (J_{i,j})$ on a network can be uniquely recovered from the knowledge of $|J| = (|J_{i,j}|)$ and the flux of the flow on the boundary nodes, where $J_{i,j}$ is the flow from node i to node j and $J_{i,j} = -J_{j,i}$. Convergent numerical algorithms for solving such problems are also presented.

1 Introduction

Let $G = (V, E)$ be a simple, undirected, weighted graph with n vertices. We can identify G with an electrical network by placing a resistor with resistance R_{ij} between every two vertices i and j , for $0 \leq i, j \leq n$ with $i \neq j$. We assign the weight $\sigma_{ij} = \frac{1}{R_{ij}}$ on each edge E_{ij} , and let $\sigma_{ij} = 0$ if i and j are not connected. Suppose a voltage is applied to a subset of the vertices, denoted by ∂V and called the boundary of V , then a current $J = (J_{ij})_{n \times n}$ will be induced on the edges of the graph, where J_{ij} is the current flowing from vertex i to vertex j . In particular, $J_{ij} = -J_{ji}$ and if the current flows from i to j , then $J_{ij} > 0$. We will also assume that $J_{ij} = 0$ if the vertices i and j are not connected by an edge, and that $J_{ii} = 0$. Note that $V = \partial V \cup \text{int}(V) = \{1, 2, \dots, n\}$. We will view the voltage potential on V as a vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ where v_i is the voltage potential at vertex i . We will also denote the imposed voltage potential on the boundary nodes by a function $f : \partial V \rightarrow \mathbb{R}$. By Kirchoff's and Ohm's Law

$$\sum_{j=1}^n \sigma_{ij}(v_i - v_j) = 0 \quad \text{for all } i \in \text{int}(V), \quad (1)$$

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where $\text{int}(V) = V \setminus \partial V$ are the interior nodes, and $u = f$ on ∂V is the imposed voltage on the boundary nodes (Dirichlet boundary condition). Assume $((\sigma_{ij})_{n \times n}, f)$ is given on $E \times \partial V$. Then (1) can be written as a system of $m = |\text{int}(V)|$ linear equations with m unknowns, i.e.

$$A_D v = b, \quad (2)$$

where v is a m dimensional column vector containing the unknown voltage values at the interior nodes, A_D is a $m \times m$ non-singular matrix (see Proposition 2.1 below) depending on the conductivities, and b is a m dimensional column vector depending on the conductivities and the known voltage at the boundary. In particular the forward problem (1) always has a unique solution which is indeed the voltage potential associated to the conductivity problem on the network.

On the other hand if a current $0 \neq g \in \mathbb{R}^{|\partial V|}$ is injected to the network on a subset of vertices $\partial V \subset V$ (Neumann boundary condition), then we necessarily have

$$\sum_{i=1}^{|\partial V|} g_i = 0, \quad (3)$$

and by Kirchoff's and Ohm's Law the voltage potential v satisfies

$$\begin{cases} \sum_{j=1}^n \sigma_{ij}(v_i - v_j) = 0 & \text{for all } i \in \text{int}(V) \\ \sum_{j=1}^n \sigma_{ij}(v_i - v_j) = g_i & \text{for all } i \in \partial V. \end{cases} \quad (4)$$

The above equation can be written as

$$A_N v = b, \quad (5)$$

where A_N is an $n \times n$ matrix depending on the conductivity $\sigma = (\sigma_{ij})_{n \times n}$, and b is an n -dimensional column vector depending on the injected current on the boundary ∂V . The matrix A_N also has unique solutions up to adding a constant (see Propositions 3.1 and 3.2 below) and the solution of (5) is the voltage potential on the vertices of the graph. The matrix A_N is in fact the well known graph laplacian of a weighted undirected graph.

As described above, the forward problems always have unique solutions up to a constant and can be easily solved by solving a linear system of equations. In this paper we are interested in the inverse problem of determining the conductivity matrix of an electrical network from the knowledge of the induced current along the edges of the network and Dirichlet or Neumann boundary conditions. This problem can also be understood as a design problem where one aims to design an electrical network that induces a prescribed current along its edges when a voltage $f \in \mathbb{R}^{|\partial V|}$ is applied to the boundary nodes ∂V , or when a current $g \in \mathbb{R}^{|\partial V|}$ is injected on ∂V . These inverse problems are in the spirit of Current Density Imaging (CDI) and Current Density Impedance Imaging (CDII) in dimensions $n \geq 2$ which have been actively studied in recent years because of its potential applications in medical imaging, see [13, 15–29]. In dimension $n = 3$ the induced current inside the conductive body Ω can be measure by Magnetic Resonance Imaging (MRI), see [13, 17].

To the authors' best knowledge the natural inverse problem considered in this paper has not been studied elsewhere. In [3] and [4], the authors investigate the problem of recovering the conductivity of the edges from the measurement of voltages at the boundary vertices, and measurements of the voltage, current, and conductivity on the boundary respectively. In [3] the authors proved injectivity of this inverse problem for critical, circular and planar graphs and provided an explicit reconstruction method. Under the assumption of monotonicity of conductivities, partial uniqueness results are established in [4]. While the general theory of inverse problems on graphs is a rich field of study with applications in various disciplines, the above results are most closely related to this work.

There is a close connection between electrical networks and random walks on graphs (see [5]). In Section 5 we exploit this connection and apply our results on electrical networks to study the inverse problem of determining transition probabilities of random walk models from the net number of times the walker passes along the edges of the graph.

The paper is organized as follows. In Section 2 we study the problem of determining the conductivity matrix of an electrical network from the knowledge of the magnitude of the induced current with Dirichlet boundary condition, and in Section 3 we study this problem with Neumann boundary data. In Section 4 we present a numerical algorithm for finding minimizers of the l^1 minimization problem we obtain in Sections 2 and 3. In Section 5 the connection between random walks and electrical networks is discussed and we apply our results on electrical networks to the inverse problem of determining transition probabilities from the net number of time a random walker passes along the edges of the graph.

2 Dirichlet Boundary Condition

In this section we study the inverse problem of determining the conductivity matrix $\sigma = (\sigma_{ij})_{n \times n}$ from the knowledge of its induced current $J = (J_{ij})_{n \times n}$ on E and the imposed voltage potential f on ∂V (Dirichlet boundary conditions). Let $G = (V, E)$ be an undirected, simple, connected graph with n vertices, and suppose a voltage is applied to some subset of the vertices inducing the current $J = (J_{ij})_{n \times n}$ on E . Throughout the paper $|J|$ denotes the matrix $|J| := (|J_{ij}|)_{n \times n}$, we will refer to $|J|$ as a measurement matrix.

We first show that the forward problem has a unique solution, i.e. A_D is non-singular. One can find a proof in [3] and we present a brief proof for the sake of completeness.

Proposition 2.1 *The matrix A_D is non-singular.*

Proof. For every $i \in \text{int}(V)$ it follows from (1) that v_i is the weighted average of the voltage potential in its neighboring nodes, i.e.

$$v_i = \frac{\sum_{j=1}^n \sigma_{ij} v_j}{\sum_{j=1}^n \sigma_{ij}}. \quad (6)$$

Consequently v satisfies the strong maximum principle in the sense that if v attains its maximum or minimum on an interior node, then v must be constant on V . In particular, v attains its minimum and maximum on the boundary ∂V .

Now suppose $A_D v_1 = A_D v_2 = b$. Then $w = v_2 - v_1$ satisfies

$$\sum_{j=1}^n \sigma_{ij}(w_i - w_j) = 0 \quad \text{for all } i \in \text{int}(V).$$

Since $w = 0$ on ∂V , it follows from the above maximum principle that $w = 0$ on V . Thus the matrix A_D is non-singular. \square

An immediate consequence of Proposition 2.1 is that the forward problem (1) always has a unique solution.

Definition 1 We say that a vertex i is an interior vertex and write $i \in \text{int}(V)$ if

$$J_i := \sum_{j=1}^n J_{ij} = 0.$$

Otherwise we say that i is boundary vertex and write $i \in \partial V$. For every $i \in \partial V$, J_i is the current flowing in ($J_i < 0$) or out ($J_i > 0$) of the graph at vertex i . In particular, $V = \text{int}(V) \cup \partial V$ and $\text{int}(V) \cap \partial V = \emptyset$.

Definition 2 Given $f : \partial V \rightarrow \mathbb{R}$ and a measurement matrix $a = (a_{ij})_{n \times n}$ with $a_{ij} \in [0, \infty)$ for all $1 \leq i, j \leq n$ and $a_{ij} = 0$ when $i = j$ and $E_{i,j} \notin E$, we say that a symmetric matrix $\sigma = (\sigma_{ij})_{n \times n}$ with $\sigma_{ij} \in [0, \infty]$ is a conductivity matrix associated to the data (f, a) , if there exists a function $v : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$ with $v|_{\partial V} = f$, and a matrix $J = (J_{ij})_{n \times n}$ such that

$$J_{ij} = \sigma_{ij}(v_i - v_j) \quad \text{and} \quad |J_{ij}| = a_{ij} \quad \text{for all } i, j \text{ with } v_i \neq v_j,$$

and

$$\sum_{j=1}^n J_{ij} = 0$$

for all $i \in \text{int}(V)$. When $a_{ij} \neq 0$ and $v_i = v_j$, then we formally define $\sigma_{ij} = \infty$ and say that the edge between nodes i and j is a perfect conductor. We shall also refer to the function v as a voltage potential and denote the set of all voltage potentials corresponding to the data (f, a) by $\mathcal{V}_{(f,a)}$.

For any measurement matrix $a = (a_{ij})_{n \times n}$, define the function $I : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$I(u) = \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j|, \tag{7}$$

and for $f \in \mathbb{R}^{|\partial V|}$ consider the minimization problem

$$\min\{I(u) : u \in \mathbb{R}^n \text{ and } u|_{\partial V} = f\}. \tag{8}$$

We shall prove that $u \in \mathcal{V}_{(f,a)}$ if and only if it is a minimizer of the least gradient problem. Let us first study the dual of the minimization problem above.

2.1 The Dual problem

Here we discuss the dual of the least gradient problem (8) and study the connection between these two problems.

Let $\mathcal{H}(V)$ be the set of all real valued functions on the vertices. We shall view a function $u \in \mathcal{H}(V)$ as a vector in \mathbb{R}^n . Also let $\mathcal{H}(E)$ to be the space of all functions on E , i.e. the space of all $n \times n$ matrices $b = (b_{ij})$, where b_{ij} denotes the value of the function on the edge from vertex i to j , with the additional convention that $b_{ij} = 0$ if the edge from i to j is not in E , and $b_{ii} = 0$.

Definition 3 *Let $u, v \in \mathcal{H}(V)$ and $a, b \in \mathcal{H}(E)$. Then we define the inner products*

$$\langle u, v \rangle_{\mathcal{H}(V)} = \sum_{i=1}^n u_i v_i, \quad \langle b, d \rangle_{\mathcal{H}(E)} = \sum_{i,j} b_{ij} d_{ij} \quad (9)$$

on $\mathcal{H}(V) \times \mathcal{H}(V)$ and $\mathcal{H}(E) \times \mathcal{H}(E)$, respectively. The spaces $\mathcal{H}(V)$ and $\mathcal{H}(E)$ equipped with the above inner products are Hilbert spaces.

Next we define two linear operators $D : \mathcal{H}(V) \rightarrow \mathcal{H}(E)$ and $\text{div} : \mathcal{H}(E) \rightarrow \mathcal{H}(V)$ which play crucial roles in our arguments.

Definition 4 *For $u \in \mathcal{H}(V)$ we define $Du \in \mathcal{H}(E)$ as*

$$(Du)_{ij} = u_i - u_j \quad (10)$$

if the edge connecting i to j is in E , and 0 otherwise. Also for $b \in \mathcal{H}(E)$ we define $\text{div} b \in \mathcal{H}(V)$ as follows

$$(\text{div} b)_i = \sum_j b_{ji} - b_{ij}. \quad (11)$$

Observe that if $b \in \mathcal{H}(E)$ is anti-symmetric, that is $b_{ij} = -b_{ji}$ for all $1 \leq i, j \leq n$, then the divergence is simply $-2 \sum_j b_{ij}$. We shall refer to D and div operators as gradient and divergence, respectively, since they play the role in our setting of the standard gradient and divergence operators on \mathbb{R}^n , $n \geq 2$. Note that the definition of the gradient and divergence given here does not depend on the weights (conductivities) of the graph as it would normally when defining these operators on a weighted graph. Since in the inverse problems we consider in this paper, the conductivities are unknown, these definitions are desirable. Let us first show that $-\text{div}$ is the adjoint of D .

Proposition 2.2 *Let $u \in \mathcal{H}(V)$ and $b \in \mathcal{H}(E)$. Then*

$$\langle u, -\text{div} b \rangle_{\mathcal{H}(V)} = \langle Du, b \rangle_{\mathcal{H}(E)}.$$

Proof. Let $u \in \mathcal{H}(V)$ and $b \in \mathcal{H}(E)$. Then

$$\begin{aligned}
\langle u, -\operatorname{div} b \rangle_{\mathcal{H}(V)} &= \sum_i u_i (-(\operatorname{div} b)_i) \\
&= \sum_i u_i \sum_j (b_{ij} - b_{ji}) \\
&= \sum_i \sum_j u_i b_{ij} - \sum_j \sum_i u_j b_{ij} \\
&= \sum_{i,j} (u_i - u_j) b_{ij} \\
&= \sum_{i,j} (Du)_{ij} b_{ij} \\
&= \langle Du, b \rangle_{\mathcal{H}(E)}.
\end{aligned}$$

□

Let $f \in \mathbb{R}^{|\partial V|}$ and define

$$\mathcal{H}_f = \{u \in \mathcal{H}(V) : u|_{\partial V} = f\}.$$

Then for $0 \leq a \in \mathcal{H}(E)$ and $f \in \mathbb{R}^{|\partial V|}$, the least gradient problem (8) can be written as

$$\min_{u \in \mathcal{H}_f} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j| = \min_{u \in \mathcal{H}_f} \frac{1}{2} \langle a, |Du| \rangle_{\mathcal{H}(E)}, \quad (12)$$

where we have used the notation $|Du|_{ij} = |(Du)_{ij}|$. Now choose $u_f \in \mathcal{H}_f$. Define $\mathcal{H}_0(V) \subset \mathcal{H}(V)$ to be the space of functions on V which are equal to zero on ∂V . Then we can equivalently write the primal problem (12) as

$$\min_{u \in \mathcal{H}_0(V)} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j + (u_f)_i - (u_f)_j| = \min_{u \in \mathcal{H}_0(V)} \frac{1}{2} \langle a, |Du + Du_f| \rangle_{\mathcal{H}(E)}. \quad (13)$$

Define $F : \mathcal{H}(E) \rightarrow \mathbb{R}$ and $G : \mathcal{H}_0(V) \rightarrow \mathbb{R}$ as follows

$$F(d) = \frac{1}{2} \langle a, |d + Du_f| \rangle_{\mathcal{H}(E)} \quad \text{and} \quad G(u) \equiv 0. \quad (14)$$

Then (13) can be written as

$$(P) \quad \alpha_P := \min_{u \in \mathcal{H}_0(V)} F(Du) + G(u).$$

By Rockafellar-Fenchel duality (see [7]), this problem admits a dual problem which can be expressed as

$$\max_{b \in \mathcal{H}(E)} -G^*(-\operatorname{div} b) - F^*(-b), \quad (15)$$

where F^* and G^* denote the convex conjugate of F and G , respectively. It is easy to see that

$$\begin{aligned} G^*(u) &= \sup_{v \in \mathcal{H}_0(V)} \sum_i u_i v_i \\ &= \begin{cases} 0 & \text{if } u \equiv 0 \text{ on } \text{int}(V) \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Next we compute the convex conjugate of F .

Lemma 2.1 *Let $a = (a_{ij}) \in \mathcal{H}(E)$ with $a_{ij} \geq 0$ and $u_f \in \mathcal{H}_f(V)$. Then*

$$F^*(b) = \begin{cases} -\langle b, Du_f \rangle_{\mathcal{H}(E)} & \text{if } |b| \leq \frac{1}{2}a \\ \infty & \text{otherwise.} \end{cases} \quad (16)$$

Proof. Suppose $|b| \leq \frac{1}{2}a$, that is $|b_{ij}| \leq \frac{1}{2}a_{ij}$ for all i, j . Then

$$\begin{aligned} F^*(b) &= \sup_{d \in \mathcal{H}(E)} (\langle d, b \rangle_{\mathcal{H}(E)} - \frac{1}{2} \langle a, |d + Du_f| \rangle_{\mathcal{H}(E)}) \\ &= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{d \in \mathcal{H}_a(E)} (\langle d, b \rangle_{\mathcal{H}(E)} - \frac{1}{2} \langle a, |d| \rangle_{\mathcal{H}(E)}) \\ &= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{d \in \mathcal{H}_a(E)} \left(\sum_{i,j} d_{ij} b_{ij} - \frac{1}{2} a_{ij} |d_{ij}| \right) \\ &\leq -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{d \in \mathcal{H}_a(E)} \sum_{i,j} |d_{ij}| (|b_{ij}| - \frac{1}{2} a_{ij}) \\ &\leq -\langle b, Du_f \rangle_{\mathcal{H}(E)}. \end{aligned}$$

Taking $d = 0$ we also get $F^*(b) \geq -\langle b, Du_f \rangle_{\mathcal{H}(E)}$.

Now suppose that there exists $1 \leq i_0, j_0 \leq n$ such that $|b_{i_0 j_0}| > \frac{1}{2}a_{i_0 j_0}$. Let $d_{i_0 j_0} = \lambda b_{i_0 j_0}$, and $d_{ij} = 0$ otherwise, where $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned} F^*(b) &= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{d \in \mathcal{H}_a(E)} \left(\sum_{i,j} d_{ij} b_{ij} - \frac{1}{2} a_{ij} |d_{ij}| \right) \\ &\geq -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{\lambda > 0} \lambda (b_{i_0 j_0}^2 - \frac{1}{2} a_{i_0 j_0} |b_{i_0 j_0}|) \\ &= -\langle b, Du_f \rangle_{\mathcal{H}(E)} + \sup_{\lambda > 0} \lambda |b_{i_0 j_0}| (|b_{i_0 j_0}| - \frac{1}{2} a_{i_0 j_0}) \\ &= \infty. \end{aligned}$$

□

Thus the dual problem (15) can be written as

$$(D) \quad \alpha_D := \sup \{ -\langle b, Du_f \rangle_{\mathcal{H}(E)} : b \in \mathcal{H}(E), |b| \leq \frac{1}{2}a, \text{ and } \text{div}(b) \equiv 0 \text{ on } \text{int}(V) \}.$$

Given that $u_i = 0$ for at least one $i \in V$ one can show that any minimizing sequence of the primal problem is uniformly bounded. Hence a convergent subsequence exists and a minimizer of the primal problem (P) always exists. On the other hand, it follows from Theorem III.4.1 in [7] that the dual problem (D) also has a solution. Indeed since $I(u) = \frac{1}{2}\langle a, |Du + Du_f| \rangle_{\mathcal{H}(E)}$ is convex and $J : \mathcal{H}(E) \rightarrow \mathbb{R}$ with $J(p) = \frac{1}{2}\langle a, |p| \rangle_{\mathcal{H}(E)}$ is continuous at $p = 0$, the condition (4.8) in the statement of Theorem III.4.1 in [7] is satisfied. The weighted l^1 minimization problem (8) does not have a unique minimizer and thus the conductivity inducing the current J on E is not unique. However we can characterize the non-uniqueness.

Theorem 2.2 *The infimum of the primal problem (P) is equal to the supremum of the dual problem (D). Moreover, the dual problem has an optimal solution b , and $J = -2b$ satisfies*

$$|J_{ij}| = a_{ij} \text{ for every } i, j \text{ with } v_i \neq v_j \quad (17)$$

and

$$J_{ij}(v_i - v_j) \geq 0 \text{ for all } 1 \leq i, j \leq n, \quad (18)$$

for every minimizer v of (8). Conversely, if $u \in \mathcal{H}_f$ and the above equation holds then u is a minimizer of (8).

Proof. A solution b to the dual problem always exists and the infimum of the primal problem (P) is equal to the supremum of the dual problem by Theorem III.4.1 in [7] as discussed above. Let v be a minimizer of (8). Then

$$\begin{aligned} \alpha_P = I(v) &= \frac{1}{2} \sum_{i,j} a_{ij} |v_i - v_j| \geq \sum_{i,j} |b_{ij}| |v_i - v_j| \geq \sum_{i,j} -b_{ij}(v_i - v_j) \\ &= \langle -b, Dv \rangle_{\mathcal{H}(E)} = \langle \operatorname{div} b, v \rangle_{\mathcal{H}(V)} \\ &= \sum_{i \in \partial V} (\operatorname{div} b)_i v_i = \sum_{i \in \partial V} (\operatorname{div} b)_i f_i = \alpha_D = \alpha_P. \end{aligned} \quad (19)$$

Hence the inequalities in 19 are indeed equalities and thus

$$|b_{ij}| = \frac{1}{2} a_{ij} \text{ for every } i, j \text{ with } v_i \neq v_j$$

and

$$b_{ij}(v_i - v_j) \leq 0 \text{ for all } 1 \leq i, j \leq n.$$

Therefore if we let $J = -2b$ we see that (17) and (18) hold. It is not hard to see that the converse also holds from the above computations. \square

Corollary 2.3 *If u and v are two arbitrary minimizers of (8), then*

$$(u_i - u_j)(v_i - v_j) \geq 0 \text{ for all } 1 \leq i, j \leq n.$$

2.2 Voltage Potentials Have Minimum Energy

We are now ready to prove the following theorem.

Theorem 2.4 *Let f be a function on ∂V and a be a measurement matrix. Then $v \in \mathcal{V}_{(f,a)}$ if and only if it is a minimizer of the least gradient problem (8).*

Proof. Suppose $v \in \mathcal{V}_{(f,a)}$ and let J be the corresponding current on E . Then

$$\begin{aligned} I(v) &= \frac{1}{2} \sum_{i,j} a_{ij} |v_i - v_j| = \frac{1}{2} \sum_{i,j} |J_{ij}| |v_i - v_j| \geq \frac{1}{2} \sum_{i,j} J_{ij} (v_i - v_j) \\ &= \sum_{i=1}^n v_i \sum_{j=1}^n J_{ij} = \sum_{i \in \text{int}(V)} v_i J_i + \sum_{i \in \partial V} v_i J_i \\ &= \sum_{i \in \partial V} v_i J_i = \sum_{i \in \partial V} f_i J_i. \end{aligned} \tag{20}$$

Therefore the minimum of the least gradient problem (8) is equal to $\sum_{i \in \partial V} f_i J_i$. Moreover the minimum is achieved for every $v \in \mathcal{V}_{(f,|J|)}$.

Now suppose v is a minimizer of the problem (8) and let b be a solution of the dual problem (D) and let $J = -2b$. Then by Theorem 2.2

$$|J_{ij}| = a_{ij} \text{ for all } i, j \text{ with } v_i \neq v_j$$

and since $\text{div} J = 0$ on $\text{int}(V)$

$$\sum_{j=1}^n J_{ij} = 0 \text{ for all } i \in \text{int}(V).$$

For $v_i \neq v_j$ define $\sigma_{ij} = \frac{J_{ij}}{v_i - v_j} \geq 0$. Then

$$J_{ij} = \sigma_{ij} (v_i - v_j) \quad \text{for all } i, j \text{ with } v_i \neq v_j.$$

Thus $v \in \mathcal{V}_{(f,a)}$ and the proof is complete. \square

Remark 2.5 *Note that every minimizer v of (8) uniquely determines a conductivity matrix σ . Corollary 2.3 indicates that the directions of the flow of the current along the edges is unique, despite multiplicity of the minimizer of (8). Indeed if two conductivity matrices σ^1 and σ^2 with $0 \leq \sigma_{ij}^1, \sigma_{ij}^2 < \infty$ induce the currents J^1 and J^2 on a network when the voltage f is imposed on ∂V , and $|J^1| = |J^2|$, then $J^1 = J^2$. This is a counter-intuitive result.*

2.3 Multiple Measurements

Suppose we have two data sets (f^1, a^1) and (f^2, a^2) , and would like to find a conductivity matrix σ inducing the currents with magnitudes a^1 and a^2 , when the voltage potentials f^1 and f^2 are imposed on the boundary vertices $\partial^1 V$ and $\partial^2 V$, respectively.

Let I^1 and I^2 be defined by Equation (7) for a^1 and a^2 respectively and for $u = (u^1, u^2) \in \mathbb{R}^n \times \mathbb{R}^n$ define

$$\Phi(u^1, u^2) = \sum_{\mathcal{C}^2} \left| \frac{u_i^1 - u_j^1}{|J_{ij}^1|} - \frac{u_i^2 - u_j^2}{|J_{ij}^2|} \right|^2, \quad (21)$$

where

$$\mathcal{C}^2 = \{(i, j) : 1 \leq i, j \leq n \text{ and } J_{ij}^1, J_{ij}^2 \neq 0\}.$$

Define

$$\mathcal{F}(u^1, u^2) = I^1(u^1) + I^2(u^2) + \Phi(u^1, u^2) \quad (22)$$

and

$$\mathcal{A} := \{(u^1, u^2) \in \mathbb{R}^n \times \mathbb{R}^n : u^1 = f^1 \text{ on } \partial^1 V \text{ and } u^2 = f^2 \text{ on } \partial^2 V\}.$$

Now consider

$$\inf_{(u^1, u^2) \in \mathcal{A}} \mathcal{F}(u^1, u^2). \quad (23)$$

It is easy to see that (23) always has a minimizer.

Theorem 2.6 *Let (u^1, u^2) be a minimizer of (23).*

1. *If there exists a conductivity matrix σ which induces the current J^i with $|J^i| = a^i$ when the voltage potential f^i is imposed on the boundary, denoted $\partial^i V$, $i = 1, 2$, then $\Phi(u^1, u^2) = 0$. Moreover,*

$$\sigma_{ij} = \frac{a_{ij}^1}{|u_i^1 - u_j^1|} \text{ for all } i, j \text{ with } u_i^1 \neq u_j^1,$$

and

$$\sigma_{ij} = \frac{a_{ij}^2}{|u_i^2 - u_j^2|} \text{ for all } i, j \text{ with } u_i^2 \neq u_j^2.$$

2. *If there doesn't exist a conductivity matrix σ inducing the current J^i with $|J^i| = a^i$ when the voltage potential f^i is imposed on the boundary noted $\partial^i V$, $i = 1, 2$, then $\Phi(u^1, u^2) > 0$.*

Proof. (1) Suppose there exists a conductivity matrix σ producing the data (f^1, a^1) and (f^2, a^2) . It follows directly from Theorem 2.4 that the set of minimizers of (23) is equal to $\mathcal{V}_{(f^1, a^1)} \times \mathcal{V}_{(f^2, a^2)}$. So the first statement follows.

(2) Suppose $\Phi(u^1, u^2) = 0$. Then u^1 and u^2 minimize I^1 and I^2 over the appropriate spaces and so by Theorem 2.4, $u^1 \in \mathcal{V}_{(f^1, a^1)}$ and $u^2 \in \mathcal{V}_{(f^2, a^2)}$ and thus they each have corresponding conductivity matrices σ^1 and σ^2 that generate currents J^1 and J^2 respectively. However $\Phi(u^1, u^2) = 0$ implies that these conductivities are in fact equal. \square

Now suppose a finite data set of measurements is given:

$$(f^1, a^1), (f^2, a^2), \dots, (f^k, a^k), \quad k \geq 2.$$

Define

$$I^l = \frac{1}{2} \sum_{ij} a_{ij}^l |u_i - u_j|, \quad 1 \leq l \leq k,$$

and

$$\Phi^k(u^1, u^2, \dots, u^k) = \sum_{l=2}^k \sum_{\mathcal{B}^l} \left| \frac{u_i^1 - u_j^1}{|J_{ij}^1|} - \frac{u_i^l - u_j^l}{|J_{ij}^l|} \right|^2,$$

where

$$\mathcal{C}^l = \{(i, j) : 1 \leq i, j \leq n \text{ and } J_{ij}^1, J_{ij}^l \neq 0\}.$$

Consider the weighted l^1 minimization problem

$$\inf_{(u^1, u^2, \dots, u^k) \in \mathcal{A}^k} \sum_{l=1}^k I^l(v^l) + \Phi^k(u^1, u^2, \dots, u^k), \quad (24)$$

where

$$\mathcal{A}^k := \{(u^1, u^2, \dots, u^k) : u^l \in \mathbb{R}^n \text{ and } u^l = f^l \text{ on } \partial^l V, \quad i = 1, 2, \dots, k\}.$$

One can similarly prove the following theorem.

Theorem 2.7 *Let (u^1, u^2, \dots, u^k) be a minimizer of (24).*

1. *If there exists a conductivity matrix σ which inducing the current J^i with $|J^i| = a^i$ when the voltage potential f^i is imposed on the boundary noted $\partial^i V$, $i = 1, 2, \dots, k$, then $\Phi(u^1, u^2, \dots, u^k) = 0$. Moreover,*

$$\sigma_{ij} = \frac{a_{ij}^l}{|u_i^l - u_j^l|} \text{ for all } i, j \text{ with } u_i^l \neq u_j^l, \quad l = 1, 2, \dots, k.$$

2. *If there doesn't exist a conductivity matrix σ inducing the current J^i with $|J^i| = a^i$ when the voltage potential f^i is imposed on the boundary noted $\partial^i V$, $i = 1, 2, 3, \dots, k$, then $\Phi(u^1, u^2, \dots, u^k) > 0$.*

3 Neumann Boundary Condition

Let $G = (V, E)$ be an undirected simple connected graph with n vertices, and suppose the current $0 \neq g \in \mathbb{R}^{|\partial V|}$ is injected to a subset ∂V of V , regarded as boundary of V , inducing the current $J = (J_{ij})$ on E . Then g should satisfy the compatibility assumption

$$\sum_{i=1}^{|\partial V|} g_i = 0. \quad (25)$$

We will again denote $|J| := (|J_{ij}|)_{n \times n}$ and refer to $|J|$ as a measurement matrix. The following proposition characterizes solutions of the forward problem (4).

Proposition 3.1 *Let A_N be the matrix defined in (5). Then*

$$\text{Ker}(A_N) = \{(c, c, \dots, c) \in \mathbb{R}^n : c \in \mathbb{R}\}.$$

Proof. Suppose $A_N w = 0$ for some $w \in \mathbb{R}^n$. Then it follows from (4) that

$$\begin{aligned} \frac{1}{2} \sum_{i,j} \sigma_{ij} (w_i - w_j)^2 &= \frac{1}{2} \sum_{i=1}^n w_i \sum_{j=1}^n \sigma_{ij} (w_i - w_j) - \frac{1}{2} \sum_{j=1}^n w_j \sum_{i=1}^n \sigma_{ij} (w_i - w_j) \\ &= \sum_{i=1}^n w_i \sum_{j=1}^n \sigma_{ij} (w_i - w_j) \\ &= 0. \end{aligned}$$

Hence $w_i = w_j$ for all i and j connected by an edge. Since G is connected the proof is complete. \square

Proposition 3.2 *The equation $A_N v = b$ has a solution if and only if $\sum_{i=1}^n b_i = 0$.*

Proof. By the Fredholm Alternative from linear algebra, $A_N v = b$ has a solution if and only if $b \in \text{Ker}(A_N^T)^\perp$. By the previous proposition and the fact that A_N is symmetric we have

$$\text{Ker}(A_N^T)^\perp = \text{Ker}(A_N)^\perp = \{b \in \mathbb{R}^N : \sum_{i=1}^n b_i = 0\}.$$

\square

Therefore if $\sum_{i=1}^n b_i = 0$, up to adding a constant the equation (4) has a unique solution. The following is the analog to Definition 2.

Definition 5 *Given $0 \neq g : \partial V \rightarrow \mathbb{R}$ satisfying $\sum_{i=1}^{|\partial V|} g_i = 0$ and a measurement matrix $a = (a_{ij})_{n \times n}$ with $a_{ij} \in [0, \infty)$ for all $1 \leq i, j \leq n$ and $a_{ij} = 0$ when $i = j$ and $E_{ij} \notin E$, we say that a symmetric matrix $\sigma = (\sigma_{ij})_{n \times n}$ with $\sigma_{ij} \in [0, \infty]$ is a conductivity matrix associated to the data (g, a) , if there exists a function $v : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$ with and a matrix $J = (J_{ij})_{n \times n}$ such that*

$$J_{ij} = \sigma_{ij} (v_i - v_j) \quad \text{and} \quad |J_{ij}| = a_{ij} \quad \text{for all } i, j \text{ with } v_i \neq v_j,$$

$$\sum_{j=1}^n J_{ij} = g_i \quad \text{for all } i \in \partial V$$

and

$$\sum_{j=1}^n J_{ij} = 0 \quad \text{for all } i \in \text{int}(V).$$

When $a_{ij} \neq 0$ and $v_i = v_j$, then we formally define $\sigma_{ij} = \infty$ and say that the edge between nodes i and j is a perfect conductor. We shall also refer to the function v as a voltage potential and denote the set of all voltage potentials corresponding to the data (g, a) by $\mathcal{V}_{(g,a)}$.

For a measurement matrix $a = (a_{ij})_{n \times n}$, define the function $I : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$I(u) = \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j|. \quad (26)$$

Also for $g \in \mathbb{R}^{|\partial V|}$ satisfying (25) define

$$\mathcal{M}_g := \{u \in \mathbb{R}^n : \sum_{i \in \partial V} u_i g_i = 1\}.$$

We shall prove that the voltage potential is a minimizer of the l^1 minimization problem

$$\min_{u \in \mathcal{M}_g} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j|. \quad (27)$$

Let us first study the dual of this problem.

3.1 The Dual problem

In this section we discuss the dual of the least gradient problem (27) and study its connection to the primal problem. Let $0 \neq g \in \mathbb{R}^{|\partial V|}$ satisfying (25). Choose $u_g \in \mathcal{H}(V)$ such that

$$\sum_{i \in \partial V} (u_g)_i g_i = 1.$$

Define

$$\mathcal{M}_0 := \{u \in \mathcal{H}(V) : \sum_{i \in \partial V} u_i g_i = 0\}.$$

Then we can equivalently write the primal problem (27) as

$$\min_{u \in \mathcal{M}_0} \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j + (u_g)_i - (u_g)_j| = \min_{u \in \mathcal{M}_0} \frac{1}{2} \langle a, |Du + Du_g| \rangle_{\mathcal{H}(E)}. \quad (28)$$

Define $F : \mathcal{H}(E) \rightarrow \mathbb{R}$ and $G : \mathcal{M}_0 \rightarrow \mathbb{R}$ as follows

$$F(d) = \frac{1}{2} \langle a, |Du + Du_g| \rangle_{\mathcal{H}(E)} \quad \text{and} \quad G(u) \equiv 0. \quad (29)$$

Then (28) can be written as

$$(P_N) \quad \alpha_{P_N} := \min_{u \in \mathcal{M}_0} F(Du) + G(u).$$

As before this problem admits a dual problem which can be expressed as

$$\max_{b \in \mathcal{H}(E)} -G^*(-\operatorname{div} b) - F^*(-b). \quad (30)$$

From Lemma 2.1 we have

$$F^*(b) = \begin{cases} -\langle b, Du_g \rangle_{\mathcal{H}(E)} & \text{if } |b| \leq \frac{1}{2}a \\ \infty & \text{otherwise.} \end{cases}$$

Next we compute G^* .

Lemma 3.1 *Let $G : \mathcal{M}_0 \rightarrow \mathbb{R}$ be defined as $G \equiv 0$. Then for $G^* : (\mathcal{M}_0)^* \rightarrow \mathbb{R}$ we have*

$$G^*(D^*b) = \begin{cases} 0 & \text{if } b \in \mathcal{B} \\ \infty & \text{otherwise,} \end{cases} \quad (31)$$

where

$\mathcal{B} := \{b \in \mathcal{H}(E) : \operatorname{div} b \equiv 0 \text{ on } \operatorname{int}(V) \text{ and } (\operatorname{div} b)_i = \lambda g_i \text{ for all } i \in \partial V, \text{ for some } \lambda \in \mathbb{R}\}.$

Proof. First note that

$$\begin{aligned} G^*(D^*b) &= \sup_{u \in \mathcal{M}_0} \langle D^*b, u \rangle_{\mathcal{H}(V)} = \sup_{u \in \mathcal{M}_0} \langle b, Du \rangle_{\mathcal{H}(E)} = \sup_{u \in \mathcal{M}_0} -\langle \operatorname{div} b, u \rangle_{\mathcal{H}(V)} \\ &= \begin{cases} 0 & \text{if } \operatorname{div} b \in \mathcal{M}_0^\perp \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Let $h \in \mathcal{H}(V)$ with $h_i = 0$ if $i \in \operatorname{int}(V)$ and $h_i = g_i$ if $i \in \partial V$, and

$$N = \{\lambda h : \lambda \in \mathbb{R}\} \subset \mathcal{H}(V).$$

Observe that $\mathcal{M}_0 = \{u \in \mathcal{H}(V) : \langle h, u \rangle_{\mathcal{H}(V)} = 0\}$. Hence $\mathcal{M}_0 = N^\perp$. Since $N^{\perp\perp} = N$, see [14],

$$\mathcal{M}_0^\perp = N,$$

and the result follows. \square

Therefore the dual problem (30) can be written as

$$(D_N) \quad \alpha_{D_N} := \sup_{b \in \mathcal{D}} \{-\langle b, Du_g \rangle_{\mathcal{H}(E)}\},$$

where $\mathcal{D} = \{b \in \mathcal{B} : |b| \leq \frac{1}{2}a\}$.

Similar to before one can show that (27) has a minimizer. Similar to the Dirichlet boundary condition case, it follows from Theorem III.4.1 in [7] that the dual problem (D_N) also has a solution and characterizes the non-uniqueness of solutions of the primal problem (27).

Theorem 3.2 *The infimum of the primal problem (P_N) is equal to the supremum of the dual problem (D_N) . Moreover, the dual problem has an optimal solution b , and $J = -2b$ satisfies*

$$|J_{ij}| = a_{ij} \text{ for every } i, j \text{ with } u_i \neq u_j \quad (32)$$

and

$$J_{ij}(u_i - u_j) \geq 0 \text{ for all } 1 \leq i, j \leq n, \quad (33)$$

for every minimizer u of (27). Conversely, if (32) and (33) hold for some \mathcal{M}_g , then u is a minimizer of (27).

Proof. Let b be a solution to the dual problem with corresponding $\lambda \in \mathbb{R}$. Suppose u is a minimizer of 27. Then

$$\begin{aligned} \alpha_{P_N} = I(u) &= \frac{1}{2} \sum_{i,j} a_{ij} |u_i - u_j| \geq \sum_{i,j} |b_{ij}| |u_i - u_j| \geq \sum_{i,j} -b_{ij} (u_i - u_j) \\ &= \langle -b, Du \rangle_{\mathcal{H}(E)} = \langle \operatorname{div} b, u \rangle_{\mathcal{H}(V)} \\ &= \lambda \sum_{i \in \partial V} g_i u_i = \lambda = \alpha_{D_N} = \alpha_{P_N}. \end{aligned} \quad (34)$$

Thus the inequalities in (34) are indeed equalities and taking $J = -2b$ we see that (32) and (33) hold. It is easy to see from the above computations that the converse also holds. \square

Corollary 3.3 *If u and v are two arbitrary minimizers of (27), then*

$$(u_i - u_j)(v_i - v_j) \geq 0 \text{ for all } 1 \leq i, j \leq n.$$

3.2 Voltage Potentials Have Minimum Energy

We can now prove the analog to Theorem 2.4.

Theorem 3.4 *Let $g \neq 0$ be a function on ∂V satisfying 25 and a be a measurement matrix. If $v \in \mathcal{V}_{(g,a)}$, then v is a minimizer of the least gradient problem (27). Conversely, given any $a = (a_{i,j})$ with $a_{i,j} \geq 0$ and $g \in \mathbb{R}^{|\partial V|}$ satisfying (25), if v is a minimizer of the least gradient problem (27), then $v \in \mathcal{V}_{(\lambda g, a)}$ for some $\lambda > 0$.*

Proof. Suppose $v \in \mathcal{V}_{(g,a)}$ and let J be the corresponding current on E . Following similar computations as in the proof of Theorem 2.4 we have

$$\begin{aligned} I(v) &= \frac{1}{2} \sum_{i,j} a_{ij} |v_i - v_j| = \frac{1}{2} \sum_{i,j} |J_{ij}| |v_i - v_j| \geq \frac{1}{2} \sum_{i,j} J_{ij} (v_i - v_j) \\ &= \sum_{i \in \partial V} v_i g_i = 1. \end{aligned} \quad (35)$$

Therefore the minimum of the least gradient problem (27) is equal to 1. Moreover the minimum is achieved for every $v \in \mathcal{V}_{(g,|J|)}$.

Now suppose v is a minimizer of the problem (27) and let b be a solution of the dual problem (D_N) with the corresponding $\lambda \in \mathbb{R}$. Let $J = -2b$. Then by Theorem 3.2 we see that $v \in \mathcal{V}_{(\lambda g, a)}$. \square

Remark 3.5 *Note that Corollary 3.3 indicates that the direction of the flow of the current along the edges is unique, despite multiplicity of the minimizers of (8) (see also Remark 2.5).*

3.3 Multiple Measurements

Suppose we have two data sets (g^1, a^1) and (g^2, a^2) , and would like to find a conductivity matrix σ inducing the currents with magnitudes $|J^1|$ and $|J^2|$, when the currents g^1 and g^2 are injected on the boundary vertices $\partial^1 V$ and $\partial^2 V$, respectively. We can consider the minimization problem

$$\inf_{(v^1, v^2) \in \mathcal{K}} F(v^1, v^2). \quad (36)$$

where F is defined by (22) and

$$\mathcal{K} := \{(v^1, v^2) \in \mathbb{R}^n \times \mathbb{R}^n : \sum_{i=1}^n v_i^1 g_i^1 = 1 \text{ on } \partial^1 V \text{ and } \sum_{i=1}^n v_i^2 g_i^2 = 1\}.$$

The analog to Theorem 2.6 can be formulated and proved in this setting and we can also similarly extend to a finite number of measurements.

4 Algorithms for finding minimizers

In this section we present numerical algorithms for finding minimizers of the l^1 minimization problems discussed in Sections 3 and 4, yielding voltage potentials for Dirichlet or Neumann boundary conditions. The primal problem (P_D) and (P_N) can be written as

$$\min_{\{u \in H, d \in \mathcal{H}(E)\}} F(d) \quad \text{subject to} \quad Du = d, \quad (37)$$

where $H = \mathcal{H}_0(V)$ for the Dirichlet case and $H = \mathcal{M}_0$ for the Neuman boundary problem. This leads to the unconstrained problem

$$\min_{\{u \in H, d \in \mathcal{H}(E)\}} F(d) + \frac{\alpha}{2} \|Du - d\|^2. \quad (38)$$

To solve the above minimization problem, we use and develop an algorithm in the spirit of the alternating Split Bregman method which was first introduced by Goldstein and Osher [12]. The Split Bregman algorithm suggests initiating the vectors b^0 and d^0 , and producing the sequences u^k , b^k , and d^k as follows

$$\begin{aligned} (u^{k+1}, d^{k+1}) &= \operatorname{argmin}_{u \in H, d \in \mathcal{H}(E)} \{F(d) + \frac{\alpha}{2} \|b^k + Du - d\|_2^2\}, \\ b^{k+1} &= b^k + Du^{k+1} - d^{k+1}, \end{aligned} \quad (39)$$

where $\alpha > 0$. Since the joint minimization problem (39) in both u and d is in general expensive to solve exactly, Goldstein and Osher [12] proposed the following Alternating Split Bregman algorithm for solving problems of type (37)

$$u^{k+1} = \operatorname{argmin}_{u \in H} \|b^k + Du - d^k\|_2^2, \quad (40)$$

$$d^{k+1} = \operatorname{argmin}_{d \in \mathcal{H}(E)} \{F(d) + \frac{\alpha}{2} \|b^k + Du^{k+1} - d\|_2^2\}, \quad (41)$$

$$b^{k+1} = b^k + Du^{k+1} - d^{k+1}. \quad (42)$$

See [2, 8, 11, 12, 31, 32] for more details. It is pointed out by Esser [8] and Setzer [32] that the above idea to minimize alternatingly was first presented for the augmented Lagrangian algorithm by Gabay and Mercier [11] and Glowinski and Marroco [10]. The resulting algorithm is called the alternating direction method of multipliers (ADMM) [9] and is equivalent to the alternating split Bregman algorithm. The convergence of ADMM in finite dimensional Hilbert spaces was established by Eckstein and Bertsekas [6]. This in particular implies convergence of the alternating split Bregman algorithm in finite dimensional Hilbert spaces. Cai, Osher, and Shen [2] and Setzer [31, 32] also independently presented convergence results for the alternating split Bregman in finite dimensional Hilbert spaces. In [22] and [23] the authors proved the convergence of the alternating split Bregman algorithm in infinite dimensional Hilbert spaces by showing that the alternating split bregman algorithm corresponds to the Douglas-Rachford splitting algorithm for the dual problem. Indeed the dual problems (15) and (30) can be written in the form

$$0 \in A(-b) + B(-b), \quad (43)$$

where $A := \partial G^* o(-\text{div})$ and $B = \partial F^*$ are maximal monotone operators on H . For a set valued operator $P : H \rightarrow 2^H$, let J_P denote its resolvent, i.e. $J_P = (Id + P)^{-1}$. Douglas-Rachford splitting algorithm states that for any initial elements x_0 and p_0 and any $\alpha > 0$, the sequences p_k and x_k generated by the following algorithm

$$\begin{aligned} x_{k+1} &= J_{\alpha A}(2p_k - x_k) + x_k - p_k \\ p_{k+1} &= J_{\alpha B}(x_{k+1}), \end{aligned} \quad (44)$$

converges to some x and p respectively. Furthermore $p = J_{\alpha B}(x)$ and p satisfies

$$0 \in A(p) + B(p).$$

Let us introduce the sequences b^k and d^k with

$$x_k = \alpha(b^k + d^k) \quad \text{and} \quad p_k = \alpha b_k.$$

Notice that both sequences b^k and d^k converge. The resolvents $J_{\alpha A}(2p_k - x_k)$ and $J_{\alpha B}(x_{k+1})$ can be computed as follows

$$J_{\alpha A}(2p_k - x_k) = \alpha(b^k + Du^{k+1} - d^k) \quad (45)$$

and

$$J_{\alpha B}(x_{k+1}) = \alpha(b^k + Du^{k+1} - d^{k+1}), \quad (46)$$

where u^{k+1} and d^{k+1} are minimizers of

$$I_1(u) = \sum_{i,j} |b_{ij}^k + (Du)_{ij} - d_{ij}^k|^2 \quad (47)$$

and

$$I_2(d) = \frac{1}{2} \sum_{i,j} a_{ij} |d_{ij} + (Du_f)_{ij}| + \frac{\alpha}{2} \sum_{i,j} |b_{ij}^k + (Du^{k+1})_{ij} - d_{ij}|^2 \quad (48)$$

over $u \in \mathcal{H}_0(V)$ for the Dirichlet problem and over $u \in \mathcal{M}_0$ for the Neumann problem, and over $d \in \mathcal{H}(E)$.

In the case of Dirichlet boundary condition the minimizer of I_1 should satisfy the Euler-Lagrange equation

$$\begin{cases} \sum_{j=1}^n (Du)_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], & \forall i \in \operatorname{int}(V) \\ u_i = 0 & \text{for all } i \in \partial V. \end{cases} \quad (49)$$

It follows from Proposition 2.1 that the above system is uniquely solvable.

In the case of Neumann boundary condition, I_1 also has a unique minimizer in \mathcal{M}_0 up to adding a constant, but identifying the solutions is more subtle. First note that if u is a minimizer I_1 in \mathcal{M}_0 , then it satisfies the Euler-Lagrange equation

$$\begin{cases} \sum_{j=1}^n (Du)_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], & \forall i \in \operatorname{int}(V) \\ \sum_{j=1}^n (Du)_{ij} = \beta g_i + [\frac{1}{2}(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], & \text{for all } i \in \partial V \end{cases} \quad (50)$$

for some $\beta \in \mathbb{R}$. Conversely for $\beta \in \mathbb{R}$, every solution of the above equation which belongs to \mathcal{M}_0 is a minimizer of I_1 . Since $\sum_{i \in \partial V} g_i = 0$ and $\sum_{i=1}^n (\operatorname{div} c)_i = 0$ for any $c \in \mathcal{H}(E)$, by Propositions 3.1 and 3.2 the system (50) has a unique solution in $\mathcal{H}(V)$ for every $\beta \in \mathbb{R}$, up to adding a constant. To identify β and find a solution of (50) in \mathcal{M}_0 , let w be a solution of

$$\begin{cases} \sum_{j=1}^n (Dw)_{ij} = 0, & \forall i \in \operatorname{int}(V) \\ \sum_{j=1}^n (Dw)_{ij} = g_i & \text{for all } i \in \partial V. \end{cases} \quad (51)$$

Then

$$\begin{aligned} 0 < \frac{1}{2} \sum_{i,j} (Dw)_{ij} &= \frac{1}{2} \sum_{i=1}^n w_i \sum_{j=1}^n (Dw)_{ij} - \frac{1}{2} \sum_{j=1}^n w_j \sum_{i=1}^n (Dw)_{ij} \\ &= \sum_{i=1}^n w_i \sum_{j=1}^n (Dw)_{i,j} \\ &= \sum_{i \in \partial V} w_i g_i. \end{aligned}$$

Hence

$$\sum_{i \in \partial V} w_i g_i > 0.$$

Now let u be a solution of

$$\sum_{j=1}^n (Du)_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], \quad \forall i \in V. \quad (52)$$

Define

$$\beta := -\frac{\sum_{i \in \partial V} u_i g_i}{\sum_{i \in \partial V} w_i g_i}.$$

Then $v = u + \beta w$ belongs to \mathcal{M}_0 and satisfies the equation (50), and hence v is the unique minimizer of I_1 over \mathcal{M}_0 , up to adding a constant.

The minimizer of I_2 can be directly computed as

$$d_{ij}^{k+1} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\} \frac{w_{ij}}{|w_{ij}|} - (Du_f)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Du_f)_{ij} & \text{if } w_{ij} = 0, \end{cases} \quad (53)$$

where $w_{ij} = (Du^{k+1})_{ij} + (Du_f)_{ij} + b_{ij}^k$.

Therefore Douglas-Rachford splitting leads to the following convergent algorithm for the Dirichlet and Neumann problems.

Algorithm 1 (Finding a minimizer of the Dirichlet Problem)

Let $\alpha > 0$, $u_f \in \mathcal{H}(V)$ with $u = f$ on ∂V and initialize $b^0, d^0 \in \mathcal{H}(E)$. For $k \geq 0$:

1. Solve

$$\begin{cases} \sum_j (Du^{k+1})_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], & \forall i \in \operatorname{int}(V) \\ u_i^{k+1} = 0 & \text{for all } i \in \partial V. \end{cases} \quad (54)$$

2. Compute d^{k+1}

$$d_{ij}^{k+1} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\} \frac{w_{ij}}{|w_{ij}|} - (Du_f)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Du_f)_{ij} & \text{if } w_{ij} = 0, \end{cases} \quad (55)$$

where $w_{ij} = (Du^{k+1})_{ij} + (Du_f)_{ij} + b_{ij}^k$.

3. Set

$$b_{ij}^{k+1} = b_{ij}^k + (Du^{k+1})_{ij} - d_{ij}^{k+1}.$$

The following proposition follows directly from the convergence of Douglas-Rachford splitting algorithm and Theorem 1.2 in [22]. See also [2, 31, 32].

Proposition 4.1 *Let u^k , b^k , and d^k be the sequences produced by the Algorithm 1. Then $u^k \rightarrow u$ and $b^k \rightarrow \frac{1}{2\alpha}J$, where u and J are solutions of the (8) and its dual problem (D), respectively. In addition $d^k \rightarrow Du$. In particular u is a voltage potential corresponding to the data (f, a) and J is the induced current with $|J| = a$.*

Algorithm 2 (Finding a minimizer of the Neumann Problem)

Let $\alpha > 0$, $v_g \in \mathcal{H}(V)$ with $\sum_{i \in \partial V} v_i g_i = 0$ and initialize $b^0, d^0 \in \mathcal{H}(E)$. Also let $w \in \mathbb{R}^n$ be a solution of (51) with $w_1 = 0$. For $k \geq 0$:

1. (a) Solve

$$\begin{cases} \sum_j (Du^{k+1})_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i], & \forall i \in \operatorname{int}(V) \\ \sum_j (Du^{k+1})_{ij} = \frac{1}{2}[(\operatorname{div} b^k)_i - (\operatorname{div} d^k)_i] & \text{for all } i \in \partial V, \end{cases} \quad (56)$$

with $u_1^{k+1} = 0$.

(b) Compute

$$\beta^{k+1} = -\frac{\sum_{i \in \partial V} u_i^{k+1} g_i}{\sum_{i \in \partial V} w_i g_i}$$

and set $v^{k+1} = u^{k+1} + \beta^{k+1} w$.

2. Compute d^{k+1}

$$d_{ij}^{k+1} = \begin{cases} \max\{|w_{ij}| - \frac{a_{ij}}{2\alpha}, 0\} \frac{w_{ij}}{|w_{ij}|} - (Dv_g)_{ij} & \text{if } w_{ij} \neq 0 \\ -(Dv_g)_{ij} & \text{if } w_{ij} = 0, \end{cases} \quad (57)$$

where $w_{ij} = (Dv^{k+1})_{ij} + (Dv_g)_{ij} + b_{ij}^k$.

3. Set

$$b_{ij}^{k+1} = b_{ij}^k + (Dv^{k+1})_{ij} - d_{ij}^{k+1}.$$

Convergence of Douglas-Rachford splitting algorithm implies the following convergence result, see Theorem 1.2 in [22] and [2, 31, 32].

Proposition 4.2 *Let v^k, b^k , and d^k be the sequences produced by the Algorithm 2. Then $v^k \rightarrow v$ and $b^k \rightarrow \frac{1}{2\alpha} J$, where v and J are solutions of the (27) and it's dual problem (D_N) , respectively. In addition $d^k \rightarrow Dv$. In particular v is a voltage potential corresponding to the data $(\lambda g, a)$ for some $\lambda \in \mathbb{R}$ and J is the induced current with $|J| = a$. Moreover λ is the optimal values of the primal and dual problems (P_N) and (D_N) , i.e. $\lambda = \alpha_{P_N} = \alpha_{D_N}$.*

5 Applications to Random Walks on Graphs

Let $G = (V, E')$ be a connected, directed, and simple graph with n nodes and consider a random walk on G . Suppose a random walker begins at node a and walks until they reach node b and if they return to a before reaching b they keep walking. Let $P = (P_{ij}) \in \mathcal{H}(E)$ be the matrix of transition probabilities, i.e. $0 \leq P_{ij} \leq 1$ is the probability of the random

walker walking from node i to node j . In particular $\sum_j P_{ij} = 1$ for all $1 \leq i \leq n$. Let W_{ij} be the expected number of times the walker walks from node i to node j before exiting the graph at node b . Note that $W_{ij} = -W_{ji}$. Can one determine transition probabilities $P = (P_{ij})$ from the knowledge of the boundary vertices $\{a, b\}$ and $W = (W_{i,j})$? In this section, among other results, we show that the answer is yes, and describe an algorithm for determining such P .

There is a close connection between electrical networks and random walks on graphs [5]. Let $G = (V, E)$ be an electrical network with conductivity matrix $\sigma = (\sigma_{ij})$, $\sigma_{i,j} \in [0, \infty)$, and let $\partial V = \{a, b\}$. Suppose a current g with $g(a) = 1$ and $g(b) = -1$ is injected to the network inducing a current J along the edges. Define

$$\sigma_i := \sum_{j=1}^n \sigma_{ij} \quad \text{and} \quad P_{ij} = \frac{\sigma_{ij}}{\sigma_i} \quad (58)$$

and assign the transition probability matrix P to the graph $G = (V, E')$. Then the net number of times the walker taking an step from node i to node j is indeed J_{ij} , i.e.

$$J = W.$$

Therefore if the boundary nodes $\partial V = \{a, b\}$ and the magnitude of expected net number of times the walker should walk along the edges of the graph is prescribed, by the method presented in Section 5, one can first find a conductivity matrix σ inducing the current $J = W$ on network and compute transition probability matrix P by (58).

The connection between random walks on graphs and electrical networks with Neumann boundary condition can be generalized to the case when $\partial V = \Gamma_a \cup \Gamma_b$ with $\Gamma_a \cap \Gamma_b = \emptyset$ and $\Gamma_a, \Gamma_b \neq \emptyset$. Let $g \in \mathbb{R}^{|\partial V|}$ with $g|_{\Gamma_a} \geq 0$ and $g|_{\Gamma_b} \leq 0$ and

$$\sum_{i \in \Gamma_1} g_i = 1 \quad \text{and} \quad \sum_{i \in \Gamma_2} g_i = -1.$$

Suppose we would like to determine a transition matrix P such that if a random walker enters the network from a vertex k in Γ_a with probability g_k , then

- they exit the network at a node $l \in \Gamma_b$ with probability $|g_l|$
- the expected net number of times they pass from vertex i to node j before exiting the network is W_{ij} , $1 \leq i, j \leq n$.

As explained above, to determine the transition matrix P it suffices to find a conductivity matrix σ inducing the current $J = W$ with Neumann data g on ∂V . Then P can be computed from (58).

Suppose $\partial V = \{a, b\}$ and consider the inverse problems of determining the transition probabilities from the relative net number of times the walker walks between the edges of the graphs, i.e. $\alpha W = (\alpha W_{i,j})$ where α is a unknown constant. Then one can determine a transition probability P by finding a conductivity matrix σ by minimizing the l^1 minimization problem (7) with $a = \alpha W$, $f(a) = 1$ and $f(b) = 0$. A transition matrix can also be obtained by minimizing (27) with the Neumann boundary condition $g(a) = 1$ and $g(b) = -1$.

Remark 5.1 *Note that in this section we assume that the conductivity matrix $\sigma = (\sigma_{ij})$ satisfies $\sigma_{i,j} \in [0, \infty)$. Indeed we do not allow perfect conductors as otherwise the probability matrix P in (58) will not be well-defined. As described in the introduction, if for a minimizer v of (8) or (27) we have $v_i = v_j$ and $|J_{i,j}| \neq 0$ for some $1 \leq i, j \leq n$, then the edge (i, j) is a perfect conductor, i.e. $\sigma_{i,j} = \infty$. If v is minimizer of (8) or (27) leading to perfect conductance on an edge, then one may look for an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $u = (u_1, u_2, \dots, u_n) := (F(v_1), F(v_2), \dots, F(v_n))$ satisfies $u_i \neq u_j$ for $i \neq j$. Note that such u will also be a minimizer of (8) or (27) and would provide a conductivity matrix σ with $\sigma_{ij} \in [0, \infty)$, and hence the transition probabilities can be computed from (58). If such increasing function F does not exist, then there exists no transition probability matrix P for which the expected number of times the walker passes along the edges is W .*

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